Invertible Algebras: Algebras with a basis consisting entirely of units

Steve Szabo
Eastern Kentucky University

(joint work with Sergio López-Permouth, Jeremy Moore and Nick Pilewski)

Lens, France July 4, 2013

Introduction

Throughout this talk R is a ring with identity and $\mathcal{A}$ is an R-algebra.

Our definition of an R-algebra only requires that for $r \in R$, $a, b \in A$, $r(ab) = (ra)b$ and not the common additional requirement that $r(ab) = a(rb)$.

We do this to allow group rings $R[G]$ and matrix rings $M_n(R)$ over a noncommutative ring R to be R-algebras.

So, in principle, the algebras considered are free as left modules and all bases considered are, in fact, left bases.

Introduction

Consider a field F and a field extension of F, E. Then any basis of E over F consists entirely of units. It can be shown if E/F is a finite degree extension, then there is a basis whose inverses also form a basis.

Let G be a group and consider the group ring $R[G]$. Clearly G is a basis consisting entirely of units.

For $n > 1$ consider $M_n(R)$. Let I_i be the identity matrix for $M_i(R)$. The following is a basis consisting entirely of units for $M_n(R)$.

- For $i \neq j$, $v_{ij} = I_n + e_{ij}$
- For $i = j \leq n-2$, $v_{ii} = \begin{pmatrix} I_i & 0 & 0 \\ 0 & 0 & I_{n-i-1} \\ 0 & 1 & 0 \end{pmatrix}$
- $v_{n-1,n-1} = I_n - e_{nn} + e_{n-1,n} + e_{n,n-1}$
- $v_{nn} = I_n$

It can be shown that the set of inverses of this basis is also a basis.

Invertible Algebras

Definition

- An *invertible basis* for $\mathcal{A}$ over R is a basis $\mathcal{B}$ such that each element of $\mathcal{B}$ is invertible in $\mathcal{A}$.
- An *invertible-2(I2) basis* for $\mathcal{A}$ over R is an invertible basis $\mathcal{B}$ such that the collection $\mathcal{B}^{-1}$ of the inverses of the elements of $\mathcal{B}$ also constitutes a basis.

If $\mathcal{A}$ has an invertible basis $\mathcal{B}$ then $\mathcal{A}$ has an invertible basis which includes 1. Specifically, for $\alpha \in \mathcal{B}$, $\mathcal{B}\alpha^{-1}$ is an invertible basis for $\mathcal{A}$ which includes 1.

Note that finite field extensions, group rings and $M_n(R)$ are examples of algebras with I2 bases.

Invertible Algebras

Definition

- An *scalar closed under inverses (SCUI) basis* for $\mathcal{A}$ over R is a basis $\mathcal{B}$ such that for all $v \in \mathcal{B}$ we have $\alpha v^{-1} \in \mathcal{B}$ for some $\alpha \in U(R)$. If $\alpha = 1$ for all $v \in \mathcal{B}$, $\mathcal{B}$, is simply a *closed under inverses (CUI) basis*.
- An *scalar closed under products (SCUP) basis* for $\mathcal{A}$ over R is a basis $\mathcal{B}$ such that for all $v, w \in \mathcal{B}$ we have $\alpha vw \in \mathcal{B}$ for some $\alpha \in U(R)$. If $\alpha = 1$ for all $v, w \in \mathcal{B}$, $\mathcal{B}$, is simply a *closed under products (CUP) basis*.

Consider the field extension of $\mathbb{C}$ over $\mathbb{R}$. An invertible basis for this field extension is $\mathcal{B} = \{1, i\}$. Notice it is actually a SCUP and SCUI basis but not CUP or CUI.

Invertible Algebras

Proposition

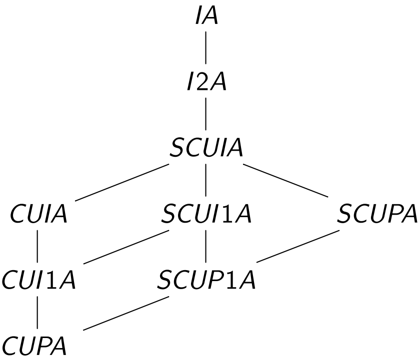
If $\mathcal{A}$ has a SCUP basis $\mathcal{B}$ over R, $\mathcal{B}$ is also a SCUI basis.

Proposition

If $\mathcal{A}$ has a CUP basis $\mathcal{B}$ over R, then $\mathcal{B}$ is group under the multiplication in $\mathcal{A}$.

Invertible Algebras

An XXX basis which includes 1 is an XXX1 basis (i.e a CUI basis with 1 is a CUI1 basis). An algebra with an XXX basis is called an XXX algebra.



Hierarchy

The inclusions in the hierarchy are almost all strict.

Consider the chain of classes, $CUPA \subset CUI1A \subset CUIA \subset SCUIA \subset I2A$.

$\frac{F_3[x,y]}{\langle x,y,xy \rangle}$ is an I2A but not a SCUIA.

The real quaternions are a SCUIA(actually a SCUP1A) but not a CUIA.

$M_2(F_2)$ is an CUIA but not a CUI1A.

$M_3(F_2)$ is an CUI1A but not a CUPA.

Hierarchy

Consider the chain of classes, $CUPA \subset SCUP1A \subset SCUPA \subset SCUIA \subset I2A$.

$\frac{F_3[x,y]}{\langle x,y,xy \rangle}$ is an I2A but not a SCUIA.

$M_2(F_2)$ is an SCUIA(actually CUIA) but not a SCUPA.

??? is an SCUPA but not a SCUP1A.

A twisted group ring that is not a group ring SCUP1A but not a CUPA.

Hierarchy

Consider the chain of classes, $CUPA \subset SCUP1A \subset SCUI1A \subset SCUIA \subset I2A$.

$\frac{F_3[x,y]}{\langle x,y,xy \rangle}$ is an I2A but not a SCUIA.

$M_2(F_2)$ is an SCUIA(actually CUIA) but not a SCUI1A.

??? is an SCUI1A but not a SCUP1A.

A twisted group ring that is not a group ring is a SCUP1A but not a CUPA.

Hierarchy: Questions

Does there exist an invertible algebra that is not a I2A?

Does there exist a SCUI1A that is not a SCUP1A?

Does there exist a SCUPA that is not a SCUP1A?

Does there exist a SCUIA that is not a SCUPA nor SCUI1 nor CUIA?

Crossed Product

Let G be a group. Then the crossed product $R * G$ is an associative ring with $\bar{G}$, a copy of G, as an R-basis. Multiplication is determined by the following two rules

1. For $x, y \in G$ there exists a unit $\tau(x, y) \in U(R)$ such that $\bar{x}\bar{y} = \tau(x, y)\overline{xy}$. This action is called the twisting of the crossed product.
2. For $x \in G$ there exists a $\sigma_x \in Aut(R)$ such that for every $r \in R$, $\bar{x}r = \sigma_x(r)\bar{x}$. This action is called the skewing of the crossed product.

When there is no twisting, a crossed product is know as a skew group ring. When there is no skewing, a crossed product is know as a twisted group ring.

Crossed Product

Just as group rings and field extensions are the archetypes of the notions of invertibility and its modifications, crossed products naturally motivated the definition of SCUP algebra as well as the following definition:

Definition

Let $\mathcal{A}$ be an invertible R-algebra. An invertible basis for $\mathcal{A}$ over R, $\mathcal{B}$, *scalarly commutes with R* if for every $v \in \mathcal{B}$ there exists some $\sigma_v \in Aut(R)$ such that for all $r \in R$ we have $vr = \sigma_v(r)v$.

Crossed Product

Proposition

$\mathcal{A}$ is a crossed product if and only $\mathcal{A}$ is an invertible R-algebra and it has an invertible basis $\mathcal{B}$ such that

1. *$\mathcal{B}$ is a SCUP basis*
2. *$\mathcal{B}$ scalarly commutes with R*

Skew Group Ring

Proposition

$\mathcal{A}$ is a skew group ring if and only $\mathcal{A}$ is an invertible R-algebra and it has an invertible basis $\mathcal{B}$ such that

1. *$\mathcal{B}$ is a CUP basis*
2. *$\mathcal{B}$ scalarly commutes with R*

Twisted Group Ring

Proposition

$\mathcal{A}$ is a twisted group ring if and only $\mathcal{A}$ is an invertible R-algebra and it has an invertible basis $\mathcal{B}$ such that

1. *$\mathcal{B}$ is a SCUP basis*
2. *$\mathcal{B}$ commutes with R*

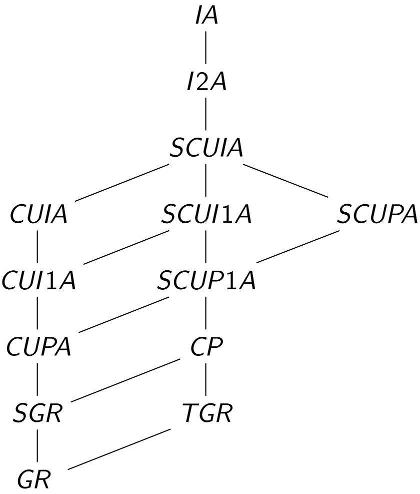
Group Ring

Proposition

$\mathcal{A}$ is a group ring if and only $\mathcal{A}$ is an invertible R-algebra and it has an invertible basis $\mathcal{B}$ such that

1. *$\mathcal{B}$ is a CUP basis*
2. *$\mathcal{B}$ commutes with R*

Expanded Hierarchy



Hierarchy: Questions

Does there exist a SCUP1A that is not a crossed product?

Does there exist a CUPA that is not a skew group ring?

Matrix Rings

Proposition

Let A be an algebra over a ring R with basis $\mathcal{B}$ such that $1 \in \mathcal{B}$. Then $M_n(A)$ is an I2 algebra over R for all $n \geq 2$.

For $1 \leq k \leq n-1$ let $P_k = I_n - e_{k,k} - e_{k+1,k+1} + e_{k,k+1} + e_{k+1,k}$, the permutation matrix that is I_n with rows k and $k+1$ interchanged.

For $b \in \mathcal{B}$ and $1 \leq i \leq n-1$ let $v_{iib} = P_i + e_{ii}b$.

For $b \in \mathcal{B} \setminus 1$ let $v_{nnb} = P_{n-1} + e_{nn}b$.

For $b \in \mathcal{B}$ and $1 \leq i,j \leq n, i \neq j$ let $v_{ijb} = I_n + e_{ij}b$.

$\mathcal{A} = \{v_{ijb}\} \cup I_n$ is an I2 basis.

Matrix Rings

Proposition

Let A be a free R-algebra with basis $\mathcal{B}$ s.t. $1 \in \mathcal{B}$. Assume n is even and $char(R) = 2$. Then $M_n(A)$ is a CUI algebra over R.

Proposition

For n even(odd) $M_n(\mathbb{F}_2)$ is a CUI(CUI1) algebra over $\mathbb{F}_2$.

Proposition

Let R be a ring of s.t. 2 is invertible in R. Then $M_n(R)$ is a CUI1 algebra over R.

Invertible not Invertible-2 Basis

Consider the F-algebra $F(x)$ of rational functions where F is an algebraic extension of a finite field.

Hou, López-Permouth, Parra(2009) showed essentially $F(x)$ consists precisely of those Laurent series that are (eventually) periodic.

Since a periodic power series is of the form $\frac{p(x)}{1-x^j} = p(x)(1 + x^j + x^{2j} + \cdots)$ for $p(x)$ a polynomial of degree less than j, where $j \in \mathbb{N}$, then periodic power series are linear combinations of elements of the form $\frac{x^i}{1-x^j}$ with $0 \leq i < j$.

It follows that eventually periodic Laurent series are generated by
$\mathcal{G} = \{x^k \mid k \in \mathbb{Z}\} \cup \{\frac{x^i}{1-x^j} \mid j \in \mathbb{Z}^+, 0 \leq i \leq j-1\}$.

Invertible-1 not Invertible-2 Basis

Notice, however, that $\mathcal{G}^{-1} \subset F[x, x^{-1}]$ (the ring of Laurent polynomials).

In particular, $\mathcal{G}^{-1}$ does not span $F(x)$.

Any basis $\mathcal{B}$ contained in $\mathcal{G}$ will be an invertible basis that is not invertible-2.

Thanks.